Translation of S4 and K5 into GF and 2VAR

Hans de Nivelle

April 10, 2001

Abstract

We give a simple, satisfiability-preserving translation of $S4$ into the intersection of the guarded fragment, and the 2-variable fragment.

1 Basics

Definition 1.1 Let F be a modal formula. An *interpretation* I of F is a structure $I = (S, R, v)$, satisfying the following:

- S is a non-empty domain.
- $R \subseteq S \times S$ is a binary relation.
- v is a function, that assigns a truth value to each atom in F in each $s \in S$. The first argument is the world, and the second the atom.

I is an $S4$-model if relation R is transitive and reflexive. We have the following (standard) definition for truth of a modal formula in an interpretation: Let F be a modal formula. We recursively define when $I = (S, R, v)$ makes F true in world $s \in S$:

- If F is an atom, then $I \models F[s]$ iff $v(s, F) = \mathbf{t}$.
- $I \models \neg A[s]$ iff not $I \models A[s]$.
- $I \models A \vee B[s]$ iff $I \models A[s]$ or $I \models B[s]$.
- $I \models A \wedge B[s]$ iff $I \models A[s]$ and $I \models B[s]$.
- $I \models \Box A[s]$ iff for every s', such that $R(s, s')$,
$$I \models A[s'].$$
- $I \models \Diamond A[s]$ iff there exists an s', such that $R(s, s')$, and
$$I \models A[s'].$$

Definition 1.2 A modal formula is in *negation normal form* (NNF) if it does not contain the $\rightarrow$, the $\leftrightarrow$, and all negations $\neg$ occur in subformulae of the form $\neg a$, where a is an atom.

Definition 1.3 A FOL-interpretation is an ordered structure $(D, [\ ])$, where D is a non-empty domain, and $[\ ]$ is a partial function. For each n-arity function symbol f, we have that $[f]$ is an n-arity function $D^n \rightarrow D$. For each n-arity predicate symbol p, $[p]$ is a subset of D^n. $I = (D, [\ ])$ is an interpretation of a formula F if I at least defines all symbols that are free in F. We write $[\ ]_d^v$ for the function that is obtained from $[\ ]$ by changing the value for v into d if $[\ ]$ was defined for v. Otherwise $[\ ]$ is extended with the value d for v.

This is all quite standard. The main unusual thing is that we do not distinguish variables from constants.

2 The Translation

Definition 2.1 Let F be a modal formula in NNF, let α and β be two first order variables. We define the following translation function $T(F, \alpha, \beta)$. The result is a first order formula, containing only the variables α and β.

- For an atom a, the translation $T(a, \alpha, \beta)$ is defined as
$$a(\alpha).$$
- For a negated atom $\neg a$, the translation $T(\neg a, \alpha, \beta)$ is defined as
$$\neg a(\alpha).$$
- For a formula of the form $A \wedge B$, the translation $T(A \wedge B, \alpha, \beta)$ equals
$$T(A, \alpha, \beta) \wedge T(B, \alpha, \beta).$$
- For a formula of the form $A \vee B$, the translation $T(A \vee B, \alpha, \beta)$ equals
$$T(A, \alpha, \beta) \vee T(B, \alpha, \beta).$$
- For a formula of the form $\Diamond A$, the translation $T(\Diamond A, \alpha, \beta)$ equals
$$\exists\beta[R(\alpha, \beta) \wedge T(A, \beta, \alpha)].$$
- For a formula of the form $\Box A$, the translation $T(\Box A, \alpha, \beta)$ equals
$$X(\alpha) \wedge \forall\alpha\beta[R(\alpha, \beta) \rightarrow (X(\alpha) \rightarrow X(\beta))] \wedge \forall\alpha[X(\alpha) \rightarrow T(A, \alpha, \beta)].$$

  Here X is a unary predicate symbol that occurs nowhere else. [1]

[1] One can define this subformula-wise, or occurrence-wise

Note the assymmetry between the handling of the $\Box$ and the $\Diamond$. Because of this T has been defined only for formulae in negation normal form. It is evident that the translation would not work for formulae that are not in negation normal form. In all the following we assume that all modal formulae that we consider are in negation normal from.

Theorem 2.2 For every modal formula F, the result $T(F, \alpha, \beta)$ is in the 2 variable fragment, and in the guarded fragment.

Proof
The only two variables used in the translation are α and β. All quantifications introduced, are guarded.

Theorem 2.3 Let F be a modal formula, that has an $S4$-model. The translation $T(F, \alpha, \beta)$ has a FOL-model. (If $\alpha \neq \beta$.)

Proof
Let $I = (S, R, v)$ be the $S4$-model of F, so there is an $s \in S$, such that

$$I \models F[s].$$

We construct the FOL-model $(S, [\ ])$, which has the same domain as I. The valuation $[\ ]$ is defined from:

- $[R] = \{(s_1, s_2) \mid R(s_1, s_2)\}$.
- For a unary predicate symbol a in $T(F, \alpha, \beta)$, that is the translation of an atom a in F, put
$$[a] = \{s \mid v(s, a) = \mathbf{t}\}.$$
- For a unary predicate symbol X in $T(F, \alpha, \beta)$, that was introduced when translating $\Box A$, put
$$[X] = \{s \mid I \models \Box A[s]\}.$$

We show by induction on the subformulae of that for all α, β and s,

$$I \models F[s] \text{ implies } (S, [\ ]_s^\alpha) \models T(F, \alpha, \beta).$$

- Let F be an atom a. If $I \models a[s]$, then $v(s, a) = \mathbf{t}$. This makes that $s \in [a]$, and so $(S, [\ ]_s^\alpha) \models a(\alpha)$.
- Let F be a negated atom $\neg a$. If not $I \models a[s]$, then $v(s, a) \neq \mathbf{t}$. This makes $s \notin [a]$, and so $(S, [\ ]_s^\alpha) \models \neg a(\alpha)$.
- Let F of the form $A \vee B$. If $I \models A \vee B[s]$, then $I \models A[s]$ or $I \models B[s]$. By induction $(S, [\ ]_s^\alpha) \models T(A, \alpha, \beta)$ or $(S, [\ ]_s^\alpha) \models T(B, \alpha, \beta)$. Then, by definition of T, it follows that $(S, [\ ]_s^\alpha) \models T(A \vee B, \alpha, \beta)$.
- Let F of the form $A \wedge B$. If $I \models A \wedge B[s]$, then $I \models A[s]$ and $I \models B[s]$. By induction $(S, [\ ]_s^\alpha) \models T(A, \alpha, \beta)$ and $(S, [\ ]_s^\alpha) \models T(B, \alpha, \beta)$. Then $(S, [\ ]_s^\alpha) \models T(A \wedge B, \alpha, \beta)$.

- Let F be of the form $\Diamond A$. Assume that $I \models \Diamond A[s]$. There exists an $s' \in S$, such that $R(s,s')$ and $I \models A[s']$. We have $(S, [\ ]_{s,s'}^{\alpha,\beta}) \models R(\alpha,\beta)$ by definition of $[R]$, and $(S, [\ ]_{s'}^{\beta}) \models T(A,\beta,\alpha)$ by induction. This together makes that
$$(S, [\ ]_s^{\alpha}) \models \exists\beta[R(\alpha,\beta) \wedge T(A,\beta,\alpha)].$$

- Let F be of the form $\Box A$. Assume that $I \models \Box A[s]$. We have to show that
$$(S, [\ ]_s^{\alpha}) \models X(\alpha),$$
$$(S, [\ ]_s^{\alpha}) \models \forall\alpha\beta[R(\alpha,\beta) \rightarrow X(\alpha) \rightarrow X(\beta)],$$
$$(S, [\ ]_s^{\alpha}) \models \forall\alpha[X(\alpha) \rightarrow T(A,\alpha,\beta)].$$
By assumption $I \models \Box A[s]$. By the construction of $[X]$, we have $s \in [X]$. This makes the first formula true.

  Assume that $(s_1, s_2) \in [R]$, and $s_1 \in [X]$, for arbitrary $s_1, s_2 \in S$. By the construction of $[X]$, we have that $I \models \Box A[s_1]$. We need to show that $s_2 \in [X]$, or that $I \models \Box A[s_2]$. Let $(s_2, s_3) \in [R]$, for an arbitrary $s_3 \in S$. Since R is transitive, we have $(s_1, s_3) \in [R]$. Then $I \models A[s_3]$. This completes the proof for the second formula.

  Assume that we have an arbitrary $s \in [X]$. By construction, it is the case that $I \models \Box A[s]$. Because $(s,s) \in [R]$, it must be the case that $I \models A[s]$. Applying induction we receive
$$(S, [\ ]_s^{\alpha}) \models T(A,\alpha,\beta).$$
But then
$$(S, [\ ]_s^{\alpha}) \models \forall\alpha[X(\alpha) \rightarrow T(A,\alpha,\beta)].$$

Next we have to show the converse:

Theorem 2.4 Let F be a modal formula. Assume that its translation $T(F,\alpha,\beta)$ has a FOL-model. Then F has an $S4$-model.

Proof
Let $(D, [\ ]) \models T(F,\alpha,\beta)$. The $S4$-model (D,R,v) has the same domain.

- First define R' from: $R'(d_1,d_2)$ iff $(d_1,d_2) \in [R]$. Let R be the transitive, reflexive closure of R'.

- For each predicate name a, put $v(d,a) = \mathbf{t}$ iff $d \in [a]$.

We show the following by induction on subformulae: If $(D, [\ ]_d^{\alpha}) \models T(F,\alpha,\beta)$, then $(D,R,v) \models F[d]$. This implies the goal, since we can put $d = [\alpha]$.

- Let F be an atom a. If $(D, [\ ]_d^{\alpha}) \models T(a,\alpha,\beta)$, then this means that $d \in [a]$. Then by construction, $v(d,a) = \mathbf{t}$. This makes $(D,R,v) \models a[d]$.

- Let F be a negated atom $\neg a$. If $(D, [\ ]_d^\alpha) \models T(\neg a, \alpha, \beta)$, this means that $d \notin [a]$. Then by construction, $v(d, a) \neq \mathbf{t}$. This makes that $(D, R, v) \models \neg a[d]$.

- Let F be of the form $A \vee B$. If $(D, [\ ]_d^\alpha) \models T(A \vee B, \alpha, \beta)$, this means that $(D, [\ ]_d^\alpha) \models T(A, \alpha, \beta)$ or $(D, [\ ]_d^\alpha) \models T(B, \alpha, \beta)$. Then by induction,

$$(D, R, v) \models A[d] \text{ or } (D, R, v) \models B[d].$$

From this it follows that

$$(D, R, v) \models A \vee B[d].$$

- Let F be of the form $A \wedge B$. If $(D, [\ ]_d^\alpha) \models T(A \wedge B, \alpha, \beta)$, this means that $(D, [\ ]_d^\alpha) \models T(A, \alpha, \beta)$ and $(D, [\ ]_d^\alpha) \models T(B, \alpha, \beta)$. Then by induction,

$$(D, R, v) \models A[d] \text{ and } (D, R, v) \models B[d].$$

From this it follows that

$$(D, R, v) \models A \wedge B[d].$$

- Let F be of the form $\diamond A$. If $(D, [\ ]_d^\alpha) \models T(\diamond A, \alpha, \beta)$, this means that $(D, [\ ]_d^\alpha) \models \exists\beta[R(\alpha, \beta) \wedge T(A, \beta, \alpha)]$. There is a $d_1 \in D$, such that $(d, d_1) \in [R]$, and $(D, [\ ]_{d_1}^\beta) \models T(A, \beta, \alpha)$. By induction, $(D, R, v) \models A[d_1]$. Since $(d, d_1) \in [R]$, we have $R'(d, d_1)$, which implies $R(d, d_1)$. But this makes that $(D, R, v) \models \diamond A[d]$.

- Finally let F be of the form $\Box A$, and assume that $(D, [\ ]_d^\alpha) \models T(\Box A, \alpha, \beta)$. This means that

$$(D, [\ ]_d^\alpha) \models X(\alpha).$$
$$(D, [\ ]_d^\alpha) \models \forall\alpha\beta[R(\alpha, \beta) \to X(\alpha) \to X(\beta)].$$
$$(D, [\ ]_d^\alpha) \models \forall\alpha[X(\alpha) \to T(A, \alpha, \beta)].$$

We need to show that $(D, R, v) \models \Box A[d]$. Assume that $R(d, d_1)$. We wish to show that $(D, R, v) \models A[d_1]$. Either there exist $e_1, \ldots, e_n = d_1$, such that

$$R'(d, e_1), R'(e_1, e_2), \ldots, R'(e_{n-1}, e_n),$$

or $d = d_1$. We show that in both cases $d_1 \in [X]$. From the first formula we know that $d \in [X]$. This proves the second case. In the other case we have

$$(d, e_1) \in [R], (e_1, e_2) \in [R], \ldots, (e_{n-1}, e_n) \in [R].$$

From the second formula we know that: If $(x_1, x_2) \in [R], x_1 \in [X]$, then $x_2 \in [X]$. From this we derive that $e_n \in [X]$. So, now in both cases we have $d_1 \in [X]$. From the third formula we know that if some $x \in [X]$, then $(D, [\ ]_x^\alpha) \models T(A, \alpha, \beta)$. This implies that $(D, [\ ]_{d_1}^\alpha) \models T(A, \alpha, \beta)$. Arrived at this point we apply induction, and we have $(D, R, v) \models A[d_1]$. This completes the proof.

3 Other Logics

The T-translation can be modified for $K4, B, T, S5$, by changing the translations for $\Box A$. We give some other translations here, the X is always a fresh name.

K Define $T(\Box, \alpha, \beta)$ as

$$X(\alpha) \wedge \forall\alpha\beta[R(\alpha, \beta) \rightarrow X(\alpha) \rightarrow T(A, \beta, \alpha)].$$

K4 Define $T(\Box A, \alpha, \beta)$ as

$$X(\alpha) \wedge \forall\alpha\beta[R(\alpha, \beta) \rightarrow X(\alpha) \rightarrow X(\beta)] \wedge \forall\alpha\beta[R(\alpha, \beta) \rightarrow X(\alpha) \rightarrow T(A, \beta, \alpha)].$$

S5 Define $T(\Box A, \alpha, \beta)$ as

$$X(\alpha) \wedge \forall\alpha[X(\alpha) \rightarrow T(A, \alpha, \beta)] \wedge \forall\alpha\beta[R(\alpha, \beta) \rightarrow X(\alpha) \rightarrow X(\beta)] \wedge$$
$$\forall\alpha\beta[R(\beta, \alpha) \rightarrow X(\beta) \rightarrow X(\alpha)].$$

4 Euclidean Frames

We obtain a similar result for the logic K5, which is the modal logic of Euclidean frames. The property of being Euclidean is the following:

Definition 4.1 A relation R is *Euclidean* if it satisfies the following property:

$$R(x, y) \wedge R(x, z) \Rightarrow R(y, z).$$

This is clearly equivalent with the following

$$R(x, y) \wedge R(x, z) \Rightarrow R(y, z) \text{ and } R(z, y).$$

Definition 4.2 Let F be a modal formula in NNF, supposed to be interpreted on a Euclidean frame. Let α and β be two distinct first order variables. We define the translation function $T(F, \alpha, \beta)$. The result is a first order formula, that contains no other variables than α and β. The translations for the classical operators are the same as for $S4$.

- For an atom a, the translation $T(a, \alpha, \beta)$ is defined as

$$a(\alpha).$$

- For a negated atom $\neg a$, the translation $T(\neg a, \alpha, \beta)$ is defined as

$$\neg a(\alpha).$$

- For a formula of the form $A \wedge B$, the translation $T(A \wedge B, \alpha, \beta)$ equals
$$T(A, \alpha, \beta) \wedge T(B, \alpha, \beta).$$
- For a formula of the form $A \vee B$, the translation $T(A \vee B, \alpha, \beta)$ equals
$$T(A, \alpha, \beta) \vee T(B, \alpha, \beta).$$
- For a formula of the form $\Diamond A$, the translation $T(\Diamond A, \alpha, \beta)$ equals
$$\exists \beta [R(\alpha, \beta) \wedge T(A, \beta, \alpha)].$$
- For a formula of the form $\Box A$, the translation $T(\Box A, \alpha, \beta)$ equals
$$\forall \beta [R(\alpha, \beta) \rightarrow T(A, \beta, \alpha)] \wedge$$
$$\forall \beta [R(\beta, \alpha) \rightarrow X(\beta)] \wedge$$
$$\forall \alpha\beta [R(\alpha, \beta) \wedge X(\alpha) \rightarrow X(\beta)] \wedge \forall \alpha\beta [R(\beta, \alpha) \wedge X(\alpha) \rightarrow X(\beta)] \wedge$$
$$\forall \alpha\beta [R(\alpha, \beta) \wedge X(\alpha) \rightarrow T(A, \beta, \alpha)].$$

Purpose of the complicated translation scheme for the $\Box A$ case is to seek application of the following lemma. The X predicate is intended to be true for the e_i.
As is the case now, the translation has exponential complexity.

Lemma 4.3 Let R be binary relation on some set D. Let $\overline{R}$ be the Euclidean closure of R. For each $d_1, d_2 \in D$, the following 2 are equivalent:

1. $\overline{R}(d_1, d_2)$,
2. Either $R(d_1, d_2)$ or there exist $e_1, \ldots, e_n \in D, (n > 1)$, with $R(e_1, d_1)$, $R(e_n, d_2)$, and for each $i, (1 \leq i < n)$, either $R(e_i, e_{i+1})$ or $R(e_{i+1}, e_i)$.

Proof
In order to show that $(\mathbf{1}) \Rightarrow (\mathbf{2})$, we write
$$\overline{R} = \bigcup_{i \geq 1} \overline{R}_i,$$
$$\overline{R}_1 = R,$$
$$\overline{R}_{i+1} = R_i \cup \{(d_1, d_2) \mid \exists e \in D, \text{ s.t. } \overline{R}_i(e, d_1) \text{ and } \overline{R}_i(e, d_2)\}.$$
We show by induction on i that $\overline{R}_i(d_1, d_2)$ implies that either $R(d_1, d_2)$, or
$$\exists e_1 \cdots e_n, \text{ s.t. } R(e_1, d_1), \ldots, R(e_i, e_{i+1}) \text{ or } R(e_{i+1}, e_i), \ldots, R(e_n, d_2).$$
If $i = 1$, then clearly $R(d_1, d_2)$. If $i > 1$, then if $R_{i+1}(d_1, d_2)$ is caused by $R_i(d_1, d_2)$, we directly apply the induction hypothesis. Otherwise there exists an

$e \in D$, such that $\overline{R}_i(e, d_1)$ and $\overline{R}_i(e, d_2)$. By applying the induction hypothesis we obtain

$$\exists f_1^1 \cdots f_{n^1}^1 \text{ s.t. } R(f_1^1, e), \ldots, R(f_i^1, f_{i+1}^1) \text{ or } R(f_{i+1}^1, f_i^1), \ldots, R(f_{n^1}^1, d_1).$$

$$\exists f_1^2 \cdots f_{n^2}^2 \text{ s.t. } R(f_1^2, e), \ldots, R(f_i^2, f_{i+1}^2) \text{ or } R(f_{i+1}^2, f_i^2), \ldots, R(f_{n^2}^2, d_2).$$

Then it is clear that

$$\begin{gathered}
\exists f_{n^1}^1 \cdots f_1^1 \; \exists e \; \exists f_1^2 \cdots f_{n^2}^2 \text{ s.t. } R(f_{n^1}^1, d_1), \ldots, \\
R(f_{i+1}^1, f_i^1) \text{ or } R(f_i^1, f_{i+1}^1), \ldots, \\
R(f_1^1, e) \text{ or } R(e, f_1^1),\; R(e, f_1^2) \text{ or } R(f_1^2, e), \ldots, \\
R(f_i^2, f_{i+1}^2) \text{ or } R(f_{i+1}^2, f_i^2), \ldots, \\
R(f_{n^2}^2, d_2).
\end{gathered}$$

This completes the first half of the proof. Next we prove $(\mathbf{2}) \Rightarrow (\mathbf{1})$. If $R(d_1, d_2)$ then it follows immediately that $\overline{R}(d_1, d_2)$. Otherwise assume that

$$\exists e_1 \cdots e_n \text{ s.t. } R(e_1, d_1), \ldots, R(e_i, e_{i+1}) \text{ or } R(e_{i+1}, e_i), \ldots, R(e_n, d_2).$$

Since $R \subseteq \overline{R}$, we have

$$\exists e_1 \cdots e_n \text{ s.t. } \overline{R}(e_1, d_1), \ldots, \overline{R}(e_i, e_{i+1}) \text{ or } \overline{R}(e_{i+1}, e_i), \ldots, \overline{R}(e_n, d_2).$$

We show that $\overline{R}(d_1, d_2)$ by induction on n, using the fact that $\overline{R}$ satisfies the axiom of being Euclidean. If $n = 1$, we have the situation that $\overline{R}(e_1, d_1)$ and $\overline{R}(e_1, d_2)$, and we can directly apply the axiom. If $n > 1$, we need to consider two cases:

- If for all $i, (1 \leq i < n)$, there is $\overline{R}(e_{i+1}, e_i)$, then we can apply the axiom on $\overline{R}(e_n, e_{n-1})$, and $\overline{R}(e_n, d_2)$, to derive $\overline{R}(e_{n-1}, d_2)$. It follows from induction that $\overline{R}(d_1, d_2)$.

- Otherwise, there is a smallest $i, (1 \leq i < n)$, such that $\overline{R}(e_i, e_{i+1})$. If $i = 1$, we have $\overline{R}(e_1, d_1)$ and $\overline{R}(e_1, e_2)$. We can apply the axiom and obtain $\overline{R}(e_2, d_1)$. It follows that

  $$\exists e_2 \cdots e_n \text{ s.t. } \overline{R}(e_2, d_1), \ldots, \overline{R}(e_i, e_{i+1}) \text{ or } \overline{R}(e_{i+1}, e_i), \ldots, \overline{R}(e_n, d_2),$$

  and we can apply induction. If $i > 1$, we have the situation that $\overline{R}(e_i, e_{i-1})$ and $\overline{R}(e_i, e_{i+1})$. Applying the axiom we obtain $\overline{R}(e_{i-1}, e_{i+1})$. So we have

  $$\begin{gathered}
  \exists e_1 \cdots e_{i-1} e_{i+1} \cdots e_n \text{ s.t. } \overline{R}(e_1, d_1), \ldots, \overline{R}(e_{i-1}, e_i) \text{ or } \overline{R}(e_i, e_{i-1}), \\
  \overline{R}(e_i, e_{i+1}) \text{ or } \overline{R}(e_{i+1}, e_i), \ldots, \overline{R}(e_n, d_2),
  \end{gathered}$$

  and we can apply induction.

This completes the proof.

Theorem 4.4 Let F be a modal formula. If F has a Euclidean model $I = (S, R, v)$, then the translation $T(F, \alpha, \beta)$ has a FOL-model.

Proof
There is an $s \in S$, such that

$$I \models F[s].$$

We construct a FOL-model $(S, [\,])$, with the same domain as I. The valuation $[\,]$ is obtained as follows:

- $[R] = \{(s_1, s_2) \mid R(s_1, s_2)\}$,
- For a unary predicate symbol a in $T(F, \alpha, \beta)$, that is the translation of an atom a in F, put
$$[a] = \{s \mid v(s, a) = \mathbf{t}\}.$$
- For a unary predicate symbol X in $T(F, \alpha, \beta)$, that was introduced for the translation of $\Box A$, put
$$[X] = \{s_1 \mid \exists s_2 \text{ s.t. } I \models \Box A[s_2] \text{ and } s_1 (R \cup R^{-1})^* R\ s_2\}.$$

We show by induction on subformulae, that for all variables α, β, and for all $s \in S$,

$$I \models F[s] \text{ implies } (S, [\,]^{\alpha}_{s}) \models T(F, \alpha, \beta).$$

- (The propositional cases are the same as in the proof of Theorem 2.3)
- Let F be of the form $\Diamond A$. Assume that $I \models \Diamond A[s]$. There exists an $s' \in S$, such that $R(s, s')$ and $I \models A[s']$. We have $(S, [\,]^{\alpha,\beta}_{s,s'}) \models R(\alpha, \beta)$, by definition of $[R]$, and $(S, [\,]^{\beta}_{s'}) \models T(A, \beta, \alpha)$, by induction. This together makes that
$$(S, [\,]^{\alpha}_{s}) \models \exists \beta [R(\alpha, \beta) \wedge T(A, \beta, \alpha)].$$
- Let F be of the form $\Box A$. Assume that $I \models A[s]$. We will to show that $(S, [\,]^{\alpha}_{s}) \models T(A, \alpha, \beta)$.
$$(S, [\,]^{\alpha}_{s}) \models \forall \beta [R(\alpha, \beta) \rightarrow T(A, \beta, \alpha)],$$
Let s_2 be arbitrary, such that $(S, [\,]^{\alpha,\beta}_{s,s_2}) \models R(\alpha, \beta)$. Then obviously $R(s, s_2)$. Since $I \models \Box A[s]$, we have $I \models A[s_2]$. By induction we see that $(S, [\,]^{\beta}_{s_2}) \models T(A, \beta, \alpha)$, so we have $(S, [\,]^{\alpha,\beta}_{s,s_2}) \models T(A, \beta, \alpha)$.
$$(S, [\,]^{\alpha}_{s}) \models \forall \beta [R(\beta, \alpha) \rightarrow X(\beta)],$$
Assume that $(S, [\,]^{\alpha,\beta}_{s,s2}) \models R(\beta, \alpha)$. Then obviously $R(s_2, s)$, so we have $s_2 (R \cup R^{-1}) R\ s$. Since $I \models \Box A[s]$, we have $s_2 \in [X]$.
$$(S, [\,]^{\alpha}_{s}) \models \forall \alpha \beta [R(\alpha, \beta) \wedge X(\alpha) \rightarrow X(\beta)] \wedge \forall \alpha \beta [R(\beta, \alpha) \wedge X(\alpha) \rightarrow X(\beta)],$$

Both members are proven analogously. Assume that for arbitrary s_1, s_2, it is the case that

$$(S, [\ ]^{\alpha,\beta}_{s_1,s_2}) \models R(\alpha,\beta) \land X(\alpha).$$

Then it is the case that $R(s_1, s_2)$, and $s_1 \in [X]$. Then by definition of $[X]$, also $s_2 \in [X]$.

$$(S, [\ ]^{\alpha}_{s}) \models \forall\alpha\beta[R(\alpha,\beta) \land X(\alpha) \rightarrow T(A,\beta,\alpha)].$$

Assume that $(S, [\ ]^{\alpha,\beta}_{s_1,s_2}) \models R(\alpha,\beta) \land X(\alpha)$, for arbitrary s_1, s_2. Then we have $R(s_1, s_2)$, and $s_1 \in [X]$. This implies that there is an $s_3 \in S$, such that $s_1(R \cup R^{-1})R\ s_3$, and $I \models \Box A[s_3]$. Because of this we have $s_2\ R^{-1}(R \cup R^{-1})^* \cup R\ s_3$. It follows from Lemma 4.3 that $R(s_3, s_2)$. This makes that $I \models A[s_2]$, and we can apply induction to obtain $(S, [\ ]^{\beta}_{s_2}) \models T(A,\beta,\alpha)$. This makes that $(S, [\ ]^{\alpha,\beta}_{s_1,s_2}) \models T(A,\beta,\alpha)$ also.

Theorem 4.5 Let F be a modal formula. If its translation $T(F,\alpha,\beta)$ has a FOL-model, then F has an K5-model.

Proof
Assume that $(D, [\,]) \models T(F,\alpha,\beta)$. The K5-model (D, R, v) has the same domain.

- First define R' from $R'(d_1, d_2)$ iff $(d_1, d_2) \in [R]$. Let R be the Euclidean closure of R'.
- For each predicate name a, put $v(d, a) = \mathbf{t}$ iff $d \in [a]$.

We show by induction on subformulae that if $(D, [\ ]^{\alpha}_{d}) \models T(F,\alpha,\beta)$, then $(D, R, v) \models F[d]$.

- (The propositional cases are the same as in the proof of Theorem 2.4)
- (The case for $\Diamond$ is the same as in the proof of Theorem 2.4)
- Finally let F be of the form $\Box A$, and assume that $(D, [\,]^{\alpha}_{d}) \models T(\Box A,\alpha,\beta)$. This means that

  A1 $(D, [\,]^{\alpha}_{d}) \models \forall\beta[R(\alpha,\beta) \rightarrow T(A,\beta,\alpha)]$,

  A2 $(D, [\,]^{\alpha}_{d}) \models \forall\alpha\beta[R(\alpha,\beta) \land X(\alpha) \rightarrow X(\beta)] \land \forall\alpha\beta[R(\beta,\alpha) \land X(\alpha) \rightarrow X(\beta)]$,

  A3 $(D, [\,]^{\alpha}_{d}) \models \forall\beta[R(\beta,\alpha) \rightarrow X(\beta)]$,

  A4 $(D, [\,]^{\alpha}_{d}) \models \forall\alpha\beta[R(\alpha,\beta) \land X(\alpha) \rightarrow T(A,\beta,\alpha)]$.

  We have to show that $(D, R, v) \models \Box A[d]$. Assume that $R(d, d_2)$. We will show that $(D, R, v) \models A[d_2]$. Using Lemma 4.3, there are two possibilities: Either $R'(d, d_2)$, or there are $e_1, \ldots, e_n, n > 0$, such that

$$R'(e_1, d), \ldots, R'(e_i, e_{i+1}) \text{ or } R'(e_{i+1}, e_i), \ldots, R'(e_n, d_2).$$

In the first case we have $(D, [\ ]^{\beta}_{d_2}) \models T(A,\beta,\alpha)$, using A1. Then by induction, we obtain $(D,R,v) \models F[d_2]$. In the second case, we have $(D, [\ ]^{\alpha,\beta}_{d,e_1}) \models R(\beta,\alpha)$, so that $(D, [\ ]^{\beta}_{r_1}) \models X(\beta)$, using A3. Successively applying A2, we obtain that

$$(D, [\ ]^{\alpha}_{e_n}) \models X(\alpha).$$

Then, since $(D, [\,]^{\alpha,\beta}_{e_n,d_2}) \models R(\alpha,\beta)$, it follows that $(D, [\,]^{\alpha,\beta}_{e_n,d_2}) \models T(A,\beta,\alpha)$, using A4. Applying induction, we see that $(D,R,v) \models A[d_2]$.

5 Acknowledgements

Thanks to Carlos Areces for proofreading a draft version.